Multiple Forecasts with Autoregressive Time Series Models: Case Studies

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Abstract: It is indisputable that accurate forecasts of economic activities are vital to successful business and government policies. In many circumstances, instead of a single forecast, simultaneous prediction intervals for multiple forecasts are more useful to decision-makers. For example, based on previous monthly sales records, a production manager would be interested in the next twelve interval forecasts of the monthly sales using for the annual inventory and manpower planning. For Gaussian autoregressive time series processes, several procedures for constructing simultaneous prediction intervals have been proposed in the literature. These methods assume a normal error distribution and can be adversely affected by departures from normality which are commonly encountered in business and economic time series. In this article, we explore the bootstrap methods for the construction of simultaneous multiple interval forecasts. To understand the mechanisms and characteristics of the proposed bootstrap procedures, several macro-economic time series are selected for illustrative purposes. The selected series are fitted reasonably well with autoregressive models which form an important class in time series. As a matter of fact, the major ideas discussed in this paper with autoregressive processes can be extended to other more complicated time series models.

Keywords: Autoregressive processes; Bootstrap; Simultaneous prediction intervals; Bonferroni inequality; AIC

1. INTRODUCTION

Multiple forecasting is often important to business planning. It provides projections for L ($L \geq 1$) consecutive future values together with the confidence intervals around these forecasts. The prediction intervals portray the optimistic and pessimistic scenarios of the underlying variable using a probability statement. These information are vital to business planning and decision.

In many applications, we are interested in generating *simultaneous* prediction intervals (i.e., based on a probability statement about all the L forecasts simultaneously), rather than a single forecast. Unfortunately, most time series computer packages do not provide comprehensive calculations for simultaneous multiple interval forecasts [see, for example, SAS Institute, 1999]. One of the main reasons is that the construction of simultaneous prediction intervals often requires intensive computation related to high-dimensional integration. In the past, the task to compute exact simultaneous multiple forecast limits for Gaussian autoregressive models is often computational infeasible for large L. Therefore, previous research efforts mainly rest on the formulation of approximate simultaneous prediction intervals [see, for examples, Ravishanker et al., 1987; Glaz and Ravishanker, 1991; Ravishanker et al., 1991]. Recently, as more advance algorithm and computational facilities are being developed, Cheung et al. [1998] introduce an 'exact method' to calculate simultaneous prediction intervals directly with reasonable computing times.

The class of linear Gaussian autoregressive models [Box and Jenkins, 1976] has been reasonably successful as a practical tool for business forecasting. However, substantial empirical evidence for non-normalities in economic time series fluctuations has been reported in the literature recently. Computing simultaneous interval forecasts based on the Gaussian assumption could be misleading. Instead of assuming a normal distribution, some authors have proposed to estimate the density of the forecast errors by bootstrap procedures. These include Thombs and Schucany [1990], Masarotto [1990],

and Cao et al. [1997]. However, their methods concentrate on the construction of the prediction interval of a single forecast. In this paper, we attempt to generalise these methods to the computation of simultaneous prediction intervals.

2. MODEL AND NOTATIONS

Suppose that a discrete time real-valued series Z_t has the stationary AR(p) representation:

$$\phi(B)Z_t = \delta + a_t \quad (1)$$

where δ is the intercept term,

$$\phi(B) = 1 - \phi_1 B - \ldots - \phi_p B^p$$

is the AR polynomial with roots outside the unit circle and have no common factors, B is the backwards shift operator such that $B^m Z_t = Z_{t-m}$, and $\{a_t\}$ is a sequence of uncorrelated random variables from a distribution F_a with mean zero and finite variance σ_a^2.

Consider a time series with available observations $\{Z_t, t = 1, \ldots, n\}$. The minimum mean square error (MMSE) forecast under model (1) at origin n for lead time l is given by

$$Z_n(l) = \delta + \phi_1 Z_n(l-1) + \cdots + \phi_p Z_n(l-p), \quad (2)$$

where $Z_n(s) = Z_{n+s}$ if $s \leq 0$. The corresponding l-step ahead forecast error $e_n(l)$ is

$$e_n(l) = Z_{n+l} - Z_n(l),$$

which has mean zero and

$$\mathrm{Var}\,[e_n(l)] = \sigma_a^2 \sum_{i=0}^{l-1} \psi_i^2. \quad (3)$$

The ψ weights are functions of the model parameters and they can be obtained recursively by $\psi_k = \sum_{i=0}^{k-1} \psi_i \phi_{k-i}$ with $\psi_0 = 1$, and $\phi_j = 0$ if $j > p$. When $\{a_t\}$ is a Gaussian white noise process, $Z_n(l)$ is normally distributed with mean Z_{n+l} and variance $\mathrm{Var}\,[e_n(l)]$. The $100(1-\alpha)\%$ prediction interval for a single l-step-ahead forecast is given by

$$Z_n(l) \pm z_{\alpha/2} \sqrt{\mathrm{Var}\,[e_n(l)]} \quad (4)$$

where $z_{\alpha/2}$ is the upper $\alpha/2$ percentage point of the standard normal distribution. In practice the model parameters $\delta, \phi_1, \ldots, \phi_p$ and σ_a^2 are not known and they are usually replaced by their estimates [Box and Jenkins, 1976, Ch. 5].

For multiple forecasting, we need to produce prediction intervals for $\{Z_{t+l}, l = 1, \ldots, L\}$ simultaneously. If the $100(1-\alpha)\%$ marginal prediction interval (4) is used for each of the forecasts, the joint confidence level may be very small, especially when L is large. Hence, we should seek for the construction of simultaneous prediction intervals of $\{Z_{n+l}, l = 1, \ldots, L\}$ such that the overall joint confidence level is controlled at a designated level $1 - \alpha$.

In the next section the existing procedures for constructing simultaneous prediction intervals for Gaussian autoregressive processes are briefly reviewed. Generalisation of the bootstrap methods by Thombs and Schucany [1990], Masarotto [1990], and Cao et al. [1997] are also presented. In Section 4 all the methods are illustrated and compared through several macroeconomic real examples. Discussion and recommendations follow in the final section.

3. CALCULATING SIMULTANEOUS PREDICTION INTERVALS

3.1 Existing Methods for Gaussian Processes

Bhansali [1974] considers a conservative method to construct the simultaneous prediction intervals, which are based on the first-order Bonferroni inequality. The $100(1-\alpha)\%$ conservative simultaneous forecast limits for $Z_{n+l}(l = 1, \ldots, L)$ are

$$Z_n(l) \pm z_{\alpha/(2L)} \sqrt{\mathrm{Var}\,[e_n(l)]}. \quad (5)$$

The intervals in equation (5) are conservative, in the sense that they provide a joint confidence level of at least $1 - \alpha$.

Under the natural ordering condition, Ravishanker et al. [1991] improve the simultaneous prediction intervals (5) using the kth-order Bonferroni inequality. In previous published work on this method, the choice of k was always less than 6 due to computational limitations. Imposing an additional condition, Glaz and Ravishanker [1991] derive another approximate method for calculating the intervals using the kth-order product-type inequality. They also show that the product-type intervals are tighter than the Bonferroni bounds. Unfortunately, the conditions required by these higher order approximation procedures are either impractical or extremely difficult to be verified [Cheung et al., 1998].

The Exact Method (EX)

Assuming Gaussian errors, the $100(1-\alpha)\%$ exact simultaneous prediction intervals for Z_{n+l}, $l = 1, \ldots, L$ are

$$Z_n(l) \pm \xi_\alpha \sqrt{\mathrm{Var}\,[e_n(l)]} \quad (6)$$

for $l = 1, \ldots, L$. Let

$$H(\xi) = Pr(|\mathcal{N}_l| \leq \xi, l = 1, \ldots, L)$$

with

$$\mathcal{N}_l = \frac{e_n(l)}{\sqrt{\mathrm{Var}\,[e_n(l)]}}$$

denoting the standardised forecast errors. Then the value of ξ_α is the solution of ξ for the following equation:

$$H(\xi) = 1 - \alpha, \tag{7}$$

where $(\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_L)$ have a multivariate normal distribution with mean vector $\mathbf{0}$ and correlation matrix $\mathbf{\Sigma} = \{\rho_{ml}\}$, $1 \leq m,\ l \leq L$. In addition, for $m \leq l$, it is easy to show that

$$\rho_{ml} = \frac{\sum_{i=0}^{m-1} \psi_i \psi_{i+l-m}}{\sqrt{\sum_{i=0}^{m-1} \psi_i^2 \sum_{j=0}^{l-1} \psi_j^2}}. \tag{8}$$

Note that if the model parameters in (1) are unknown, they will be replaced by the estimates and $\mathbf{\Sigma}$ can then be computed numerically using (8).

Given α and $\mathbf{\Sigma}$, we may use the secant method to solve the equation (7) for ξ_α. However, it requires the evaluation of an L-dimensional multivariate normal probability (i.e., the $H(\xi)$ function) many times. For large L, it was computational infeasible in the past. Recently, Cheung et al. [1998] propose an efficient algorithm to perform such calculations. The required computing times for their method are fairly acceptable (less than 30 CPU seconds) for L as large as 20. In this paper, we shall employ their methodology.

3.2 Modified Bootstrap Methods

The procedures described in the previous section assume that the errors have a Gaussian distribution. However, non-normality in residuals is often detected in economic time series model fitting. To tackle the problem, Peters and Freedman [1985] derive a procedure which is based on the bootstrap of the residuals. Li and Maddala [1996] provide an excellent review of bootstrap methods applied to linear time series analysis.

Several authors employed the bootstrap method to generate prediction limits for a *single* forecast. Masarotto [1990] (henceforth MAS) presents a bootstrap method using the empirical distribution of the residuals. Fixing the model parameters, bootstrap samples of the standardised prediction errors can be obtained. Then, the prediction interval is computed using the ordered bootstrap replications. Masarotto [1990] proves that this interval is consistent, in the sense that it gives a correct asymptotic coverage probability under some reasonable assumptions.

Thombs and Schucany [1990] (henceforth TS) consider a bootstrap method to construct the prediction interval of a single forecast from an autoregressive model. Bootstrap replicates generated backward in time are used to ensure that the probability distribution for future values of the process is conditional on the past observations. Furthermore, model parameters are re-estimated in each bootstrap replicate. It allows for estimation uncertainty when parameters are estimated by ordinary least squares (OLS).

Cao et al. [1997] (henceforth CAO) propose a simplification of the Thombs and Schucany's method. The model parameters are only estimated once and these estimates are used in the computation of all bootstrap forecasts. Also, the CAO procedures only resample the future values. Although this method does not account for the uncertainty due to parameter estimation, Cao et al. [1997] claim that it still provides satisfactory results most of time with a substantial reduction in computing efforts as compared to the TS procedure.

In this section we shall generalise the MAS, TS and CAO methods to construct simultaneous prediction intervals for an autoregression by applying the first-order Bonferroni inequity. For the MAS, TS and CAO procedures, the order of the AR model p is assumed to be known. However, in practice, the model order is seldom known. Following Grigoletto [1998], we shall use the Akaika Information Criterion (AIC) to determine the model order for a given time series. In other words, the order p will be substituted by $\hat{p}$ which minimises

$$\mathrm{AIC}(p) = n \log \hat{\sigma}_p^2 + 2p \tag{9}$$

where $\hat{\sigma}_p^2$ is the estimate of σ_a^2 based on an AR(p) model, $p = 0, 1, \ldots, P$ and P is a pre-assigned upper limit of the order of the autoregression. Note that the values of $\hat{p}$ based on the original series and each of bootstrap replications may be different. The detailed steps of each generalised bootstrap method are outlined in what follows.

The Generalised MAS Method (GMAS)

1. Obtain the least square estimates $(\hat{\delta}, \hat{\phi}_1, \ldots, \hat{\phi}_p)$.
2. Compute the future values $\hat{Z}_n(l)$, $l = 1, 2, \ldots, L$.
3. Construct $\hat{F}_a$ by computing the centred and rescaled forward residuals:

$$\hat{a}_i' = \left(\frac{n-p}{n-2p}\right)^{1/2} \left(\hat{a}_i - \frac{1}{n-p} \sum_{j=p+1}^{n} \hat{a}_j\right),$$

where

$$\hat{a}_i = Z_i - \hat{\delta} - \hat{\phi}_1 Z_{i-1} - \cdots - \hat{\phi}_p Z_{i-p},$$

for $i = n, n-1, \ldots, p+1$.

4. Generate a bootstrap replicate

$$\{Z_1^*, \ldots, Z_n^*, Z_{n+1}^*, \ldots, Z_{n+L}^*\}$$

from

$$Z_s^* = \hat{\delta} + \hat{\phi}_1 Z_{s-1}^* + \cdots + \hat{\phi}_p Z_{s-p}^* + \hat{a}_s^{'*},$$

for $s = 1, 2, \ldots, n+L$ where $\{\hat{a}_s^{'*}\}$ are bootstrap errors from $\hat{F}_a$.

5. Compute $(\hat{\delta}^*, \hat{\phi}_1^*, \ldots, \hat{\phi}_p^*)$, $\hat{\sigma}_a^{2*}$ and $\hat{\psi}_k^*$'s using the bootstrap sample $\{Z_1^*, \ldots, Z_n^*\}$. and compute the bootstrap future values $\hat{Z}_n^*(l)$ for $l = 1, 2, \ldots, L$ by

$$\begin{aligned}\hat{Z}_n^*(l) &= \hat{\delta}^* + \hat{\phi}_1^* \hat{Z}_n^*(l-1) + \cdots \\ &\quad + \hat{\phi}_p^* \hat{Z}_n^*(l-p),\end{aligned}$$

where $\hat{Z}_n^*(s) = Z_{n+s}^*$ if $s \leq 0$.

6. Compute $r_n^*(l)$ by

$$r_n^*(l) = \frac{Z_{n+l}^* - \hat{Z}_n^*(l)}{\left(\hat{\sigma}_a^{2*} \sum_{i=0}^{l-1} \hat{\psi}_i^{*2}\right)^{1/2}},$$

for $l = 1, 2, \ldots, L$.

7. Repeat Step (4) to (6) until B sets of bootstrap replicates $r_n^*(l)$ for $l = 1, 2, \ldots, L$ are obtained

8. The $100(1-\alpha)\%$ simultaneous forecast limits for $\{Z_{n+l}\}$, $l = 1, 2, \ldots, L$ are given by

$$\left[\hat{Z}_n(l) + \left(\hat{\sigma}_a^2 \sum_{i=0}^{l-1} \hat{\psi}_i^2\right)^{1/2} r_n^{*(q)}(l),\right.$$

$$\left.\hat{Z}_n(l) + \left(\hat{\sigma}_a^2 \sum_{i=0}^{l-1} \hat{\psi}_i^2\right)^{1/2} r_n^{*(B+1-q)}(l)\right],$$

where $q = \left\lceil B\left(\frac{\alpha}{2L}\right) + 1\right\rceil$ and $\{r_n^{*(1)}(l), \ldots, r_n^{*(B)}(l)\}$ are the ordered bootstrap replications of the standardized forecast errors for $l = 1, 2, \ldots, L$.

The Generalised TS Method (GTS)

1. Obtain the least square estimates $(\hat{\delta}, \hat{\phi}_1, \ldots, \hat{\phi}_p)$.

2. Compute centred and rescaled backward residuals:

$$\hat{e}_i' = \left(\frac{n-p}{n-2p}\right)^{1/2} \left(\hat{e}_i - \frac{1}{n-p} \sum_{j=1}^{n-p} \hat{e}_j\right),$$

where

$$\hat{e}_i = Z_i - \hat{\delta} - \hat{\phi}_1 Z_{i+1} - \cdots - \hat{\phi}_p Z_{i+p},$$

for $i = n-p, n-p-1, \ldots, 1$. Let $\hat{F}_e$ be the empirical distribution of the centred and rescaled backward residuals.

3. Randomly draw $\{\hat{e}_i^{'*},\ i = 1, \ldots, n-p\}$ from $\hat{F}_e$ and obtain the backward bootstrap replicate $\{Z_1^*, \ldots, Z_n^*\}$ where $Z_i^* = Z_i$, for $i = n-p+1, \ldots, n$, and $Z_i^* = \hat{\delta} + \hat{\phi}_1 Z_{i+1}^* + \cdots + \hat{\phi}_p Z_{i+p}^* + \hat{e}_i^{'*}$, for $i = 1, \ldots, n-p$.

4. Fit the backward bootstrap replicate to an AR(p) model and obtain the least square estimates $(\hat{\delta}^*, \hat{\phi}_1^*, \ldots, \hat{\phi}_p^*)$.

5. Compute centred and rescaled forward residuals:

$$\hat{a}_i' = \left(\frac{n-p}{n-2p}\right)^{1/2} \left(\hat{a}_i - \frac{1}{n-p} \sum_{j=p+1}^{n} \hat{a}_j\right),$$

where

$$\hat{a}_i = Z_i - \hat{\delta} - \hat{\phi}_1 Z_{i-1} - \cdots - \hat{\phi}_p Z_{i-p},$$

for $i = n, n-1, \ldots, p+1$. Let $\hat{F}_a$ be the empirical distribution of the centred and rescaled forward residuals.

6. Draw forward bootstrap errors $\{\hat{a}_j^{'*},\ j = n+1, n+2, \ldots, n+L\}$ from $\hat{F}_a$ and compute recursively the bootstrap future values: $Z_{n+l}^* = \hat{\delta}^* + \hat{\phi}_1^* Z_{n+l-1}^* + \cdots + \hat{\phi}_p^* Z_{n+l-p}^* + \hat{a}_{n+l}^{'*}$, for $l = 1, 2, \ldots, L$.

7. Repeat Step 3 to 6 B times.

8. The $100(1-\alpha)\%$ conservative simultaneous prediction intervals for $\{Z_{n+l}\}$, $l = 1, 2, \ldots, L$ would be

$$\left[Z_{n+l}^{*(q)},\ Z_{n+l}^{*(B+1-q)}\right],$$

where $q = \left\lceil B\left(\frac{\alpha}{2L}\right) + 1\right\rceil$ and $(Z_{n+l}^{*(1)}, \ldots, Z_{n+l}^{*(B)})$ are the ordered bootstrap future values for $l = 1, 2, \ldots, L$. The $\lceil \cdot \rceil$ denotes the greatest integer function.

The Generalised CAO Method (GCAO)

This is a simplification of the GTS method. Steps 2 to 4 of GTS are skipped and the parameters in step 6 are not re-estimated.

4. CASE STUDIES

In this section we apply all these procedures to several macro-economic time series. The first series (Series A) is the quarterly US GDP growth rates (DG_t). It is defined as the logarithmic difference of the real (in 1996 constant dollars), seasonally adjusted US GDP series from 1947.1 to 2000.4 giving 215 observations in total. An AR(1) process is fitted to the data: $DG_t = .5647 + .3410 DG_{t-1} + a_t$ with $\hat{\sigma}_a = .9625$. Series B is the quarterly Australian Money Supply (M3) growth rate series from 1971.4 to 2001.1 ($n = 118$). The series (MG_t) is in current prices and seasonally adjusted. An AR(1) process is fitted to the data: $MG_t = 1.4322 + .4819 DG_{t-1} + a_t$ with $\hat{\sigma}_a = 1.3849$. Series C is the quarterly growth rate of Melbourne/Westpac Leading Index of Economic Activity (LG_t) from 1971.4 to 2001.1 ($n = 118$). An AR(2) process is fitted to the data:

Table 1. Four Quarterly Interval Forecasts (in per cent) using Various Methods

Series	l	Gaussian Method	Bootstrap Methods		
		EX	GTS	GCAO	GMAS
A	1	(-1.70, 3.07)	(-2.05, 3.74)	(-2.06, 3.24)	(-2.19, 3.42)
	2	(-1.72, 3.32)	(-1.70, 4.08)	(-1.90, 4.18)	(-1.58, 4.22)
	3	(-1.70, 3.37)	(-1.76, 3.96)	(-1.67, 4.05)	(-1.94, 4.55)
	4	(-1.69, 3.39)	(-1.93, 4.12)	(-1.94, 4.27)	(-1.77, 4.30)
B	1	(-0.43, 6.39)	(-0.56, 9.75)	(-0.59, 9.78)	(-0.61, 9.40)
	2	(-0.92, 6.66)	(-0.91, 9.33)	(-0.93, 9.15)	(-1.06, 9.63)
	3	(-1.05, 6.68)	(-1.05, 9.27)	(-0.91, 9.16)	(-1.20, 8.53)
	4	(-1.10, 6.68)	(-1.09, 9.28)	(-1.03, 9.35)	(-1.09, 9.63)
C	1	(-2.91, 4.52)	(-2.56, 3.75)	(-5.74, 3.62)	(-6.19, 3.84)
	2	(-3.67, 5.24)	(-3.13, 5.05)	(-4.71, 4.48)	(-5.52, 4.51)
	3	(-3.90, 5.30)	(-6.78, 5.25)	(-6.66, 4.50)	(-7.38, 4.92)
	4	(-3.97, 5.27)	(-4.37, 4.56)	(-5.40, 4.55)	(-6.14, 4.65)
D	1	(-5.54, 4.85)	(-6.81, 5.24)	(-5.50, 5.21)	(-5.79, 5.44)
	2	(-4.38, 7.46)	(-4.89, 7.84)	(-4.95, 7.55)	(-5.07, 7.72)
	3	(-5.61, 6.63)	(-6.43, 6.89)	(-5.65, 6.85)	(-6.10, 7.28)
	4	(-5.11, 7.25)	(-5.63, 8.07)	(-5.61, 7.27)	(-6.16, 7.51)

$LG_t = .2908 + .6611LG_{t-1} - .1311LG_{t-2} + a_t$ with $\hat{\sigma}_a = 1.5199$. Series D is the quarterly Australian Government Final Consumption Expenditure (General) growth rate series (CG_t). The series is in current prices, seasonally adjusted from 1971.4 to 2001.1 giving a total of 118 observations. A first-order autoregression is fitted to the data: $CG_t = 1.3476 - .5454CG_{t-1} + a_t$ with $\hat{\sigma}_a = 2.1111$. Four quarterly simultaneous prediction intervals for these series by various methods are computed at the 95% confidence level. The results are given in Table 1.

Boxplots for the standardised residuals from each fitted models are displayed in Figure 1. A boxplot is a way to look at the overall shape of a set of data. The central box shows the data between the 'hinges' (roughly quartiles), with the median represented by a line. 'Whiskers' go out to the extremes of the data, and very extreme points are shown by themselves. Formal tests for normality by Jarque and Bera [1980] are applied to the residuals. The test statistics are 21.14 (0.0000) for Model A, 107.03 (0.0000) for Model B, 3.89 (0.1430) for Model C, 0.80 (0.6703) for Model D, with the corresponding p-value given in parenthesis. Residuals from the fitted AR(1) model of Series D are fairly close to normal. The widths of the intervals obtained from the EX method and the bootstrap methods are very similar (the ratios range from 86% to 98%).

For Series A, the empirical distribution of the residuals is quite symmetric, with several outlying values observed at both ends (see Figure 1). The bootstrap methods adopt these non-normal characteristics and produce prediction intervals with wider widths (but still fairly symmetrical around the forecasts) as compared to those obtained from the EX method. The width ratios are ranging from 78% to 90%.

Figure 1 shows that the residuals obtained from the fitted model for Series B are asymmetric, with extreme values concentrated on the positive end. From Table 1, we can see that the bootstrap methods automatically respond to this special feature of the error distribution and give higher (as compared to the EX method, which assumes a normal error distribution) upper limits for the forecasts. The width ratios are ranging from 66% to 79%.

For Series C, there is one large negative outlier in the residuals. The bootstrap methods have made allowance for the extreme value. Their lower bounds are much lower than those calculated from the Gaussian EX method. The width ratios (the EX method to the bootstrap methods) range from 74% to 118%.

The case studies in this section show that non-normal residuals are not uncommonly encountered in economic time series modelling. The bootstrap methods proposed in this article respond reasonably well to different non-Gaussian situations. On the contrary, simultaneous prediction intervals produced by the EX method under the normal assumption could be misleading in many practical cases.

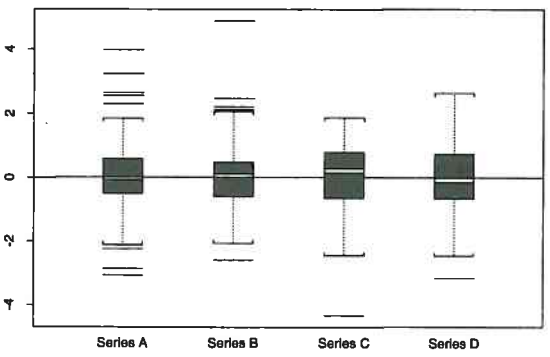

Figure 1. Boxplots for the standardised residuals

5. DISCUSSION AND RECOMMENDATIONS

Clements and Taylor [2001] consider a procedure to bias-correct the parameter estimates via bootstrap. By simulation, they show that bootstrap methods applied with a correction for estimation bias has significant improvement on the coverage probability when the sample size is small (say, $n \leq 50$). However, in practice, we seldom make long-range (i.e., large L) multiple interval forecasts using a short time series. Therefore, we did not consider their method in this paper.

Au [2000] compares the methods of constructing simultaneous prediction intervals discussed in this article through a simulation study. His conclusions are summarised as follows: (1) For non-Gaussian error situation, the EX method is not recommended; (2) For the EX procedure, the experimental joint coverage percentages are less than the nominal level even in the Gaussian cases. It might due to the parameter uncertainty problem as discussed in Chatfield [1993]; (3) The MCAO method, being a simplification version of the GTS procedure, surprisingly performed better than the GTS method in most non-normal cases considered in the experiment; and (4) The GMAS procedure is preferred for all kinds of distributions, especially the normal mixture one.

Based on Au's simulation experiment and the empirical experience in this article, we recommend the GMAS method for constructing simultaneous prediction intervals. It does not depend on the Gaussian assumption. The method also accommodates quite well the model and parameter uncertainties.

Even though our discussion in this paper mainly concerns autoregressive processes, our proposed ideas of generalisation can be easily adopted by other bootstrap methods for models with moving-average or nonlinear components [see, e.g., Pascual et al., 2001].

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